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On the topology of the Hénon map

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Abstract. Topological invariants of the Hénon map are investigated by means of the pruning front. First, a long sequence of primary homoclinic tangencies is computed, confirming the monotonicity of the front. An algorithm to extract forbidden sequences is then introduced and discussed. Forbidden sequences of increasing lengths are used to construct a hierarchy of regular grammars, represented by directed graphs, which approximate the exact grammar arbitrarily well. The topological entropy is estimated as the largest eigenvalue of their adjacency matrix. It exhibits an exponential convergence towards the asymptotic value with an exponent in agreement with a previous conjecture based on the growth rate of the number of forbidden words.

Symbolic dynamics of unimodal maps of the interval is fairly well understood. For many years it has been recognized that a binary generating partition can be introduced by splitting the interval into two subsets lying to the left and right of the maximum c_0, respectively. As a consequence, nearly all trajectories are unambiguously encoded by infinite strings of bits $S(x) = (s_0 s_1 s_2 \ldots)$ where s_i is either 0 or 1, depending whether $f^i(x)$ is $\leq c_0$ or $> c_0$, respectively [1, 2]. This allows one to study the resulting chaotic behaviour by means of the theory of formal languages [3]. As a matter of fact, the grammar of a map with a generic chaotic attractor is not trivial in that it is characterized by an infinite number of (irreducible) forbidden words. Also, the set of allowed words cannot in general be summarized by regular expressions; hence the induced grammar is not regular. Nevertheless, it is relatively simple. The kneading sequence K (i.e. the forward symbol sequence of the maximum) contains enough information to determine the grammatical rules and, in turn, the topological entropy of the map. More precisely, we first define a transformation $\tau : S \to \tau(S) \equiv (t_1, t_2, \ldots)$ for any binary sequence S, where

$$t_k = t_{k-1} + s_k \pmod 2 \qquad k = 1, 2, \ldots \tag{1}$$

with $t_0 = 0$.

In a slight abuse of notation, we shall use the same name for a semi-infinite sequence $(t_1, t_2, \ldots)$ and for the real number $\in [0, 1]$ whose binary representation is $0 \cdot t_1 t_2 \ldots$. Then the evolution of a point x can be monitored in three different ways

$$\tau(S(f(x))) = \tau(\sigma(S(x))) = T(\tau(S(x))) \tag{2}$$

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where σ denotes the shift operator and T is the tent map

$$T(\tau)=\begin{cases}2\tau & \text{if } \tau\in[0,\frac{1}{2}]\\ 2(1-\tau) & \text{if } \tau\in[\frac{1}{2},1].\end{cases} \tag{3}$$

Since the transformation to τ-values preserves the natural ordering of the real line, $\tau(K)$ turns out to be maximal, i.e. $\tau(\sigma^m(K))\leqslant\tau(K)$ for all $m\geqslant 0$. From this it follows that there is at most one irreducible forbidden word of any length. If $t_k=0$, then $(s_1,\ldots,1-s_k)$ is forbidden, while no irreducible word of length k exists if $t_k=1$ [1, 2]. This can easily be turned into algorithms for constructing the grammars explicitly [4, 5].

The complexity of chaotic dynamics is much more pronounced already for the most natural 2D extension of a unimodal map, the Hénon map $(x,y)\to(1+y-ax^2,bx)$. It is now proven [6] that there is a set of positive measure in the parameters a and b for which the map has a strange attractor. Unfortunately, the proof covers only very small values of b, and does not include the values $a=1.4$, $b=0.3$, which have been mostly studied since the original paper by Hénon [7] and will also be considered here.

The problem of constructing a 'good' binary partition for this system was practically solved in [8], where the authors pointed out that the most natural generalization of the critical point of unimodal maps is obtained by considering all 'primary' tangencies (PT) between stable and unstable manifolds (see also [4]). Denote by $C(x)$ the sum of the curvatures of the stable and unstable manifold in x. A tangency x is called primary if $C(x)\leqslant C(f^n(x))$ for all n. For $a=1.4$, $b=0.3$, the PTs are close to the x axis (see figure 1).

Subsequently, it was proposed [9] to extend the 'τ scheme' to the 2D case. With each point with $S=(\ldots s_{-1}s_0s_1\ldots)$, one associates a 'forward' variable $\tau(S)=(t_1,t_2,\ldots)$ given by (1) and a 'backward' variable $\delta(S)\equiv(d_1,d_2,\ldots)$ with

$$d_k=d_{k-1}+(1-s_{1-k})\pmod 2 \qquad k=1,2,\ldots \tag{4}$$

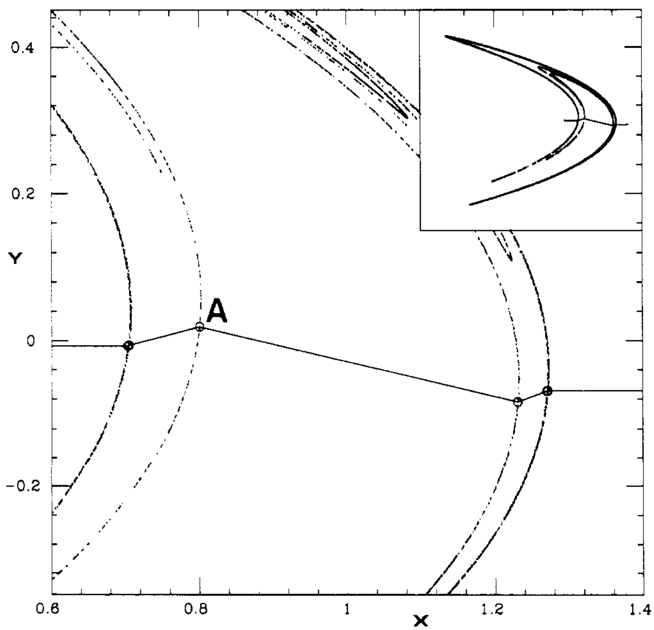

Figure 1. The binary partition of the Hénon attractor. The circles represent altogether 500 primary homoclinic tangencies. The entire Hénon attractor is shown in the inset.

with $d_0 = 1$. Application of the original map to the point (x, y) is equivalent to $(\tau, \delta) \to (T(\tau), D_\tau(\delta))$, with T given in (3) and

$$D_\tau(\delta) = \begin{cases} \frac{1}{2}(1-\delta) & \text{if } \tau \in [0, \frac{1}{2}] \\ \frac{1}{2}(1+\delta) & \text{if } \tau \in [\frac{1}{2}, 1]. \end{cases} \tag{5}$$

Analogously to unimodal maps, where all topological properties can be retrieved from the kneading sequence (the maximal τ), it has been conjectured in [9] that the grammar of 2D maps can be extracted from the so-called 'pruning front'. To each primary tangency P are attributed two symmetrical values $\delta_+(P)$ and $\delta_-(P) = 1 - \delta_+(P)$ (since s_0 is undetermined), and a kneading sequence $\tau(P)$. For all allowed points with $\delta \in [\delta_-(P), \delta_+(P)]$, τ should be less than $\tau(P)$, and thus the pruning front is obtained by cutting out rectangles $\{\tau, \delta \mid \tau > \tau(P), \delta \in [\delta_-(P), \delta_+(P)]\}$ for all P. The union of these rectangles and of their images and pre-images give the set of forbidden points (see figure 2). Notice that this picture can be correct only if the pruning front is monotonic in the half plane $\delta < \frac{1}{2}$.

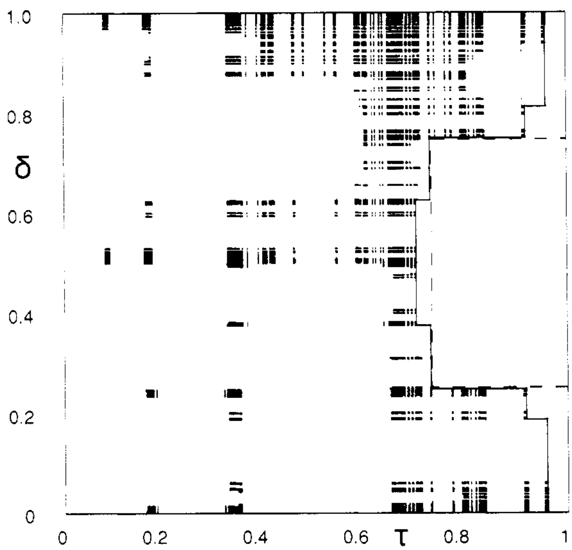

Figure 2. The symbol (τ, δ) plane of the Hénon map for $(a, b) = (1.4, 0.3)$. The pruning front is drawn as a full line. The broken line represents the rectangle cut out by the tangency point 'A' in figure 1.

Therefore, both to define a generating partition and to understand the underlying grammar, it is of crucial importance to provide accurate estimates of many homoclinic tangencies. To this end, we computed up to circa 500 tangencies by following two independent approaches, which are briefly summarized below. On segments of the unstable manifold close to the fixed point of the Hénon attractor, we selected points whose nth iterates fall close to the suspected border of the partition. By further iterating i times ($i = 1, 2, \ldots$) such points, we can determine those ones exhibiting contraction along the unstable manifold, with a rate smaller than a pre-assigned value $m_{\min}$ depending on i. By simultaneously decreasing $m_{\min}$ and increasing i, it is possible to reach any desired accuracy. Alternatively, besides the local contraction rate, we can compute the curvature of the unstable manifold [10] and thus determine the points corresponding to local maxima after i iterates. We again increase i until these points become stable, i.e. until their variation with i becomes smaller than the prescribed accuracy. These two methods did indeed give the same set of PTs, with an accuracy of 10^{-16}.

As a next step we have verified that the corresponding points of the pruning front are monotonically ordered as conjectured in [9]. This seems to be true even in a more strict sense: the τ and δ ordering is exactly equivalent to the ordering in the real line. This implies of course that the PTs can be joined by a relatively smooth polygon with no hang-overs and no sharp bends. We conjecture that this property holds down to arbitrarily small length scales, and is responsible for the success of the method.

More precisely, we found that the length of the polygon connecting all PTs is finite ($L = 0.587\,038\,57$), implying that it is not a fractal curve. It is minimal if the points are ordered as above. This also allows another definition of PTs: the set of PTs is a subset of all homoclinic tangencies such that (a) every other tangency is a (pre-)image of a PT, and (b) the length of the polygon connecting them is minimal.

Let us now briefly describe the algorithm used to extract the irreducible forbidden sequences. In the 1D case we have a single τ, each '0' in which gives rise to a pruned sequence, independently of the past bits of a given trajectory. In the Hénon map, we have to add information on the past. Corresponding to rectangles in the (τ, δ) plane (see figure 2), forbidden words are obtained by concatenating the forbidden 'future' sequence with the corresponding past. Technically, this is done by first ordering the tangency points (τ_j, δ_j) such that $\delta_j \geqslant \delta_{j-1}$ and $\tau_j > \tau_{j-1}$. Assume that the expansions of δ_j and δ_{j-1} first differ at the mth bit, and that τ_j and τ_{j-1} differ for the first time at the nth bit. Assume further that the 'zero' bits of τ_j occur at positions $n_1, n_2, \ldots$. Then, the words forbidden by (τ_j, δ_j) are just

$$(s_{-m}, s_{-m+1}, \ldots, s_{n_k-1}, 1 - s_{n_k}) \tag{6}$$

with $n_k \geqslant n$.

We have checked this procedure in the Lozi map, where the binary partition is trivial so that we can compare this method with that presented in [11]. From this analysis it turns out that the pruning front procedure works well apart from some cases where the shortest forbidden sequences are not found immediately. For instance, in the Lozi map with $a = 1.6$ and $b = 0.4$ we found that 001010010 is forbidden and irreducible. But its 'sister' 101010010 is forbidden too, as it contains the forbidden word 1010100 as a substring. Therefore, the shorter sequence 01010010 obtained by deleting the first bit is actually forbidden.

Notice that this means that we can miss some forbidden words of length $\leqslant n$, if we use the pruning front for searching only for words up to this length. Accordingly, we conclude that the pruning front gives the correct answer in the infinite-length limit, but it can provide slightly wrong finite-length estimates. This is in contrast to the method given in [11].

For the Hénon map, we have been able to estimate all (184 in total) forbidden words up to length 31. From this, we constructed the grammars which forbid exactly all words of length l by building the corresponding deterministic directed graphs [3]. For increasing l, this gives increasingly fine approximations to the exact grammar of the Hénon map. After minimization [3], the largest graph (for $l = 31$) had 676 nodes.

The growth of the number of orbits with their length is governed by the topological entropy

$$h = \lim_{n\to\infty} \frac{\log N_n}{n}. \tag{7}$$

The same growth rate holds for periodic points. Thus, the topological entropy is one of the most important characteristics of a chaotic system. In principle, there are several methods to estimate it. A comparison between some of them is reported in figure 3.

From the very definition (7) one can extrapolate the asymptotic value, using for N_n either the number of periodic points [12] (crosses in figure 3) or the total number of allowed sequences (diamonds). The latter are easily determined from our directed graphs by iterating the adjacency matrices [13]. Let us denote by $\mathbf{A}^{(l)}$ the adjacency matrix of the graph forbidding words of length up to l. Then

$$N_n = \sum_k (\mathbf{A}^{(l)n})_{1k}. \tag{8}$$

for $n \leqslant l$.

A more sophisticated method is provided by determining the leading zero $z = \mathrm{e}^{-h}$ of the inverse topological ζ-function [14], which can be written in the form

$$1/\zeta(z) = \prod_p (1 - z^{n_p}) = 1 - \sum_{k=1}^{\infty} c_k z^k \tag{9}$$

where the product extends over all primitive cycles. Their length is denoted by n_p.

We should point out that there exists a close relationship between $\zeta(z)$ and the adjacency matrices $\mathbf{A}^{(l)}$. Consider the functions

$$1/\zeta^{(l)}(z) = \det(1 - z\mathbf{A}^{(l)}) = \exp[\mathrm{tr}\log(1 - z\mathbf{A}^{(l)})] = \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathbf{A}^{(l)}{}_n) \right). \tag{10}$$

By expressing the traces in terms of prime cycles, it is straightforward to realize that

$$\zeta(z) = \lim_{l \to \infty} \zeta^{(l)}(z). \tag{11}$$

This connection provides an alternative way to compute the first coefficients c_k of ζ^{-1}, and thus to check the enumeration of periodic orbits in [12]. We should, however, point out that care has to be taken since a new forbidden word of length n can forbid a cycle of length $n_p < n$. Notice that the coefficients c_k in (9) are known if and only if

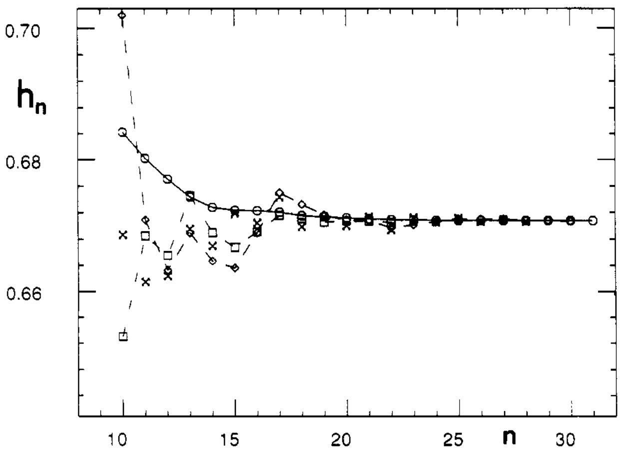

Figure 3. Topological entropy (in bits) obtained from the largest eigenvalue of graphs (circles), from the leading zero of truncated versions of $1/\zeta$ (squares) and from estimates $h_n = \log_2 N_n/n$ with N_n being either the number of periodic points (crosses) or the number of allowed sequences (diamonds), of length n.

all cycles of period $\leqslant k$ are enumerated. Thus, the coefficients $c_k^{(l)}$ of $1/\zeta^{(l)}$ have to change if a new cycle of length $\leqslant k$ is found, and agreement between c_k and $c_k^{(l)}$ can, in principle, be arbitrarily delayed. Nevertheless, we found that $c_k = c_k^{(31)}$ for $k \leqslant 23$ (for the algorithm used to compute $c_k^{(31)}$, see below).

It was proposed in [15] to estimate $1/\zeta$ by simply truncating the sequence in (9) after c_k, if all cycles are known up to length k and not beyond. This gives the squares in figure 3. We see that it yields hardly any improvements over the naive methods, in agreement with what was found in [16] for the Ruelle zeta function.

In the last approach we have estimated the topological entropy from the largest eigenvalue of the adjacency matrices [13] associated with the directed graphs mentioned above (circles in figure 3). More precisely,

$$h_l = \log \lambda_1^{(l)} \tag{12}$$

where $\lambda_1^{(l)}$ is the largest eigenvalue of $\mathbf{A}^{(l)}$. Numerically, this is easily obtained by using (8) with $n \gg l$. Figure 3 shows that this method not only yields exact upper bounds on h, but also the best convergence to the asymptotic value h. As the last method always gives upper bounds on h, it is natural to study its convergence by plotting $\log(h_{l-1} - h_l)$ against l. This is shown in figure 4, where a reasonable exponential convergence is indeed seen.

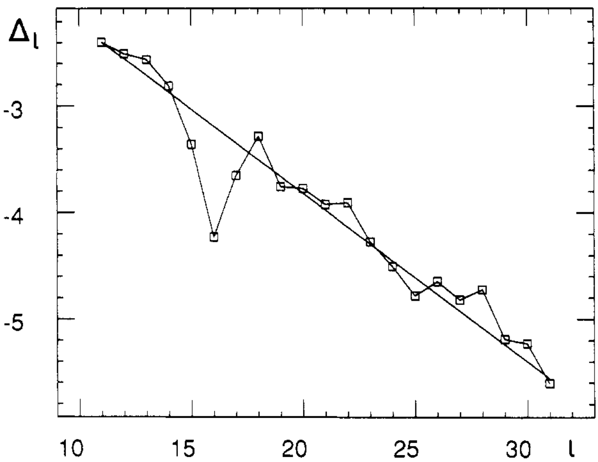

Figure 4. $\Delta_l \equiv \log(h_{l-1} - h_l)$ against l, where l represents the maximum length of forbidden sequences taken into account. The slope of the straight line is h/D (see text).

To explain such a behaviour we have extended to 2D maps a result known for unimodal maps. In [5], it has been shown that h_l converges in the latter case exponentially with an exponent given by the topological entropy itself, $h_l = h + \mathcal{O}(e^{-lh})$. This is easily understood from the fact [2] that $c_k = \pm 1$ for unimodal maps, which in turn is related to the observation that there is at most one forbidden word of any given length.

The main difference between the 1D and 2D case concerns the number of forbidden words $N_f(l)$ of given length l. In [11] it has been conjectured that

$$N_f(l) \simeq \exp\left(\frac{h(D-1)l}{D}\right) \tag{13}$$

where D is the dimension of the attractor (multifractal corrections are neglected). If we assume that the order of magnitude of the variation of the topological entropy due to the addition of a new forbidden word of length l depends only on l, then the analysis of the 1D case suggests that this contribution should be e^{-hl}. By summing over all irreducible disallowed sequences with the same length we obtain

$$h_{l-1} - h_l \simeq \exp\left(\frac{+hl}{D}\right). \tag{14}$$

The slope of the straight line in figure 4 is h/D, as predicted from (14). This good agreement confirms our previous considerations. We see that the convergence is lowered with respect to the 1D case, but it is still exponential. It appears reasonable to assume that the exponent in (14) controls the convergence of finite estimates of metric properties as well.

It is interesting to understand the growth rate of forbidden words in terms of the pruning front. In the 1D case, it is just a straight line and it cuts away a single box in the (δ, τ) plane. The growth rate in 2D maps should depend on the j-dependence of τ_j. More precisely, it should be strictly related to the fractal dimension of the set $\{\tau_j | j = 1, 2, \ldots\}$ obtained by projecting the points of the pruning front onto the τ axis.

Let κ denote the number of equal bits in the future of two homoclinic tangencies, whose distance in real space is ε. Thus

$$\varepsilon\, e^{\kappa\lambda_+} \simeq \mathcal{O}(1) \tag{15}$$

where λ_+ is the positive Lyapunov exponent. Moreover, if two such tangencies have n bits in common in the past, it is also true that

$$\varepsilon\, e^{-n\lambda_-} \simeq \mathcal{O}(1) \tag{16}$$

where λ_- is the negative exponent. As a consequence, κ can be expressed as a function of n

$$\kappa = -\frac{\lambda_-}{\lambda_+}\, n. \tag{17}$$

Now, passing to the symbol plane, if we want to cover the projection onto the τ axis with boxes of size $\varepsilon_\tau = 2^{-\kappa}$, we need e^{hn} such boxes. Combining everything together and using the Kaplan–Yorke relation, we obtain

$$D_\tau = \frac{h(D-1)}{\log 2}. \tag{18}$$

This yields $D_\tau \simeq 0.18$ for the Hénon map with standard parameter values. A direct numerical simulation performed with a box-counting algorithm and 500 PTs, yields $D_\tau = 0.177$. Equation (18) can be immediately related to (13). Indeed, the maximum length of forbidden words which can be reached by observing κ bits in the past and n bits in the future, is $l = k + n$. Thus, by expressing the number e^{hn} in terms of l rather than ε we recover (13).

Finally, let us discuss the behaviour of the zeta function, and of its approximations $\zeta^{(l)}(z)$. At variance with the 1D case, both the size of the adjacency matrices $\mathbf{A}^{(l)}$ and the coefficients c_k increase exponentially. This makes their computation more cumbersome, despite the fact that the matrices are very sparse. Indeed, the number of nodes in the lth graph (and thus the size of $\mathbf{A}^{(l)}$) should increase asymptotically with the same exponent as the number of forbidden words of length $\leqslant l$.

The most effective approach we have found is to evaluate the coefficients of $\det(1 - z\mathbf{A}^{(l)})$ by computing the cycles of the graph. It is well known (see for instance

[5]) that each closed self-avoiding path of length m on the graph contributes with -1 to the mth coefficient. More generally, any combination of r non-overlapping such cycles with total length k adds a term $(-1)^r$ to the coefficient c_k. These are the only contributions to the determinant. We have computed all the cycles up to length 31, by starting from the original graph and progressively removing one node at each step paying the price of increasing the number of links among the remaining nodes. Each time the removal of a node yields a closed loop we obtain a new cycle; if we obtain a self-touching walk we discard it and finally, if the link is too long we discard it as well (this is the only way to keep the number of links as small as possible, avoiding uncontrollable explosions). The results for the first 30 coefficients in the Hénon map are reported in table 1 for the graphs of order 30 and 31, and compared to the coefficients c_k of $1/\zeta(z)$ obtained by the method of [12]. The agreement between $c_k^{(31)}$ and c_k indicates that our list of forbidden words correctly yields all cycles of length $\leqslant 23$.

We see that the c_k grow very slowly. It seems plausible that their growth rate is also related to the growth rate of forbidden words, but it is impossible from our results to perform a reliable numerical check of this hypothesis. In order to obtain a better

Table 1. List of number $N_f(k)$ of forbidden words of length k, and of coefficients $c_k^{(30)}$, $c_k^{(31)}$ and c_k.

k	$N_f(k)$	$c_k^{(30)}$	$c_k^{(31)}$	c_k
1	0	1	1	1
2	0	1	1	1
3	0	−1	−1	−1
4	3	1	1	1
5	0	−1	−1	−1
6	0	1	1	1
7	2	3	3	3
8	3	1	1	1
9	3	−3	−3	−3
10	0	−1	−1	−1
11	2	3	3	3
12	5	−1	−1	−1
13	3	5	5	5
14	2	−5	−5	−5
15	2	−3	−3	−3
16	2	5	5	5
17	5	9	9	9
18	4	−3	−3	−3
19	4	−5	−5	−5
20	8	3	3	3
21	8	−1	−1	−1
22	6	−7	−7	−7
23	6	17	17	17
24	10	−1	−2	−3
25	7	−11	−10	−11
26	11	15	16	19
27	12	−5	−6	−7
28	22	−13	−12	−13
29	12	17	16	—
30	22	−7	−8	—
31	20	—	−22	—

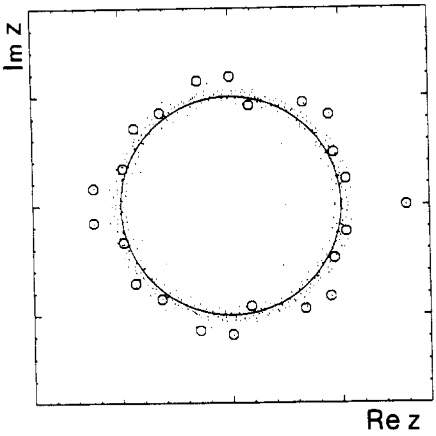

Figure 5. Eigenvalues of the graph of order 31 (dots) compared with zeros of the inverse ζ-function truncated at the 23rd coefficient. The unit circle is drawn for the sake of reference. Eigenvalues very close to the origin are not drawn because their determination was numerically unstable.

understanding of the situation, we report in figure 5 the eigenvalues of the graph (dots) corresponding to forbidden sequences up to length 31, compared with the zeros of the inverse ζ-function (circles) truncated at the 23rd coefficient. We see that the largest ones agree, as we should expect. The majority of eigenvalues concentrate at $|z| \approx 1$, and they are not related to zeros of $1/\zeta$.

Our final conclusion is that both the generating partition of [8] and the idea of a pruning front proposed in [9] have passed extremely precise numerical tests. Also the ideas put forward in [11] were verified substantially. The most easy and precise understanding of the topological dynamics was not through periodic cycles, but by directly studying the homoclinic tangency points defining the pruning front. Once this is understood, the transition to a description in terms of periodic cycles is possible and interesting.

One of our main results is a very precise and not too cumbersome estimate of the topological entropy. Such estimates might turn out to be essential if one wants to understand the dependence of the Hénon attractor on the parameters a and b. We also expect that similar methods might be useful in analysing other chaotic systems. A detailed analysis of the Lozi map shall appear elsewhere [17].

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